Reduction Based Creative Telescoping for Summation of D-finite Functions

The Lagrange Identity Approach

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Applications of Computer Algebra

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The problem of symbolic summation

Let $F_n(x) = (-1)^n n^2 J_{2n}(x)$ and $S(x) = \sum_{n=1}^{\infty} F_n(x)$.

Given mixed-differential equations satisfied by $F_n(x)$:

$$-2x(2n+1)(n+1)^2\partial_x(F_n) + n^2x^2F_{n+1} + (n+1)^2(8n^2 - x^2 + 4n)F_n = 0$$
$$n^2(n+1)(2n+1)x^2F_{n+2} + (\ldots)F_{n+1} + x^2(n+1)(2n+3)(n+2)^2F_n = 0,$$

compute the minimal linear differential equation satisfied by S:

$$x^2\partial_x^2(S) - 2x\partial_x(S) + (x^2+2)S = 0.$$

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Applications:

- Computation of closed forms
- Verification of identities
- Efficient numerical approximation of sums

Examples of identity verifications

- An identity between binomials

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \text{ (Strehl, 1994)}$$

$$(n+2)^3a(n+2)-(2n+3)(17n^2+51n+39)a(n+1)+(n+1)^3a(n)=0$$

- Legendre's generating series

$$\sum_{n=0}^{+\infty}P_n(x)z^n=(1-2xz+z^2)^{-1/2}$$

$$(2xz-z^2-1)\partial_z(y)+(x-z)y=0$$

- An Identity between special functions (here Bessel functions)

$$J_0(z\sqrt{1-u^2})=\sum_{n=0}^{\infty}\frac{(4n+1)(2n)!j_{2n}(z)P_{2n}(u)}{2^{2n}(n!)^2} \text{ (Abramowitz/Stegun)}$$

$$z\partial_z^2(y)+\partial_z(y)+z(1-u)y=0$$

$$(-u^2+1)\partial_u(y)+zu\partial_z(y)=0$$

Creative Telescoping for summation[1]

$F_n(x)$ D-finite to be summed

Goal : find r, $\lambda_i \in \mathbb{Q}(x)$ independent of n, and a function G such that

$$\underbrace{(\lambda_r(x)\partial_x^r + \cdots + \lambda_1(x)\partial_x + \lambda_0)}_{\text{telescoper}} F_n(x) = \underbrace{G(n+1,x) - G(n,x)}_{\text{G called certificate}}.$$

[1] Zeilberger, Takayama, Chyzak, Koutschan, Chen ...

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After summation w.r.t n we get:

$$(\lambda_r(x)\partial_x^r + \cdots + \lambda_1(x)\partial_x + \lambda_0) \sum_{n=0}^{N} F_n(x) = \underbrace{G(N+1,x) - G(0,x)}_{\text{often equals 0}}.$$

$\rightsquigarrow$ Generalises to sums with more parameters and any Ore operator.

[1] Zeilberger, Takayama, Chyzak, Koutschan, Chen ...

Algo1: Chyzak's algorithm (2000)

Recall $F_n(x) = (-1)^n n^2 J_{2n}(x)$ and $S(x) = \sum_{n=1}^{\infty} F_n(x)$.

Fix an order r and use an Ansatz:

$$\sum_{i=0}^{r} \lambda_i(x) \partial_x^r (F_n) = \Delta_n \left(\sum_{i,j} a_{i,j}(n,x) \partial_x^i (F_{n+j}) \right)$$

where $\Delta_n(f) = f(n+1) - f(n)$.

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$$\left(\sum_{i=0}^{r} (\ldots)\lambda_i(x)\right) F_n(x) + \left(\sum_{i=0}^{r} (\ldots)\lambda_i(x)\right) F_{n+1}(x) = \Delta_n \left(a_0(n,x)F_n(x) + a_1(n,x)F_{n+1}(x)\right)$$

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where $\Delta_n(f) = f(n+1) - f(n)$.

All computations done we get a system of recurrences with parametric rhs:

$$(\ldots)a_1(n+1,x) - a_0(n,x) = \sum_{i=0}^{r}(\ldots)\lambda_i(x)$$

$$a_0(n+1,x) + (\ldots)a_1(n+1,x) - a_1(n,x) = \sum_{i=0}^{r}(\ldots)\lambda_i(x)$$

To conclude uncouple it and find rational solutions.

Algo2: Koutschan's fast heuristic (2010)

💡 Guess the denominators Q_i in the Ansatz and avoid uncoupling

$$\sum_{i=0}^{r} \lambda_i(x)\partial_x^r(F_n) = \Delta_n \left(\sum_{i=0}^{N} \frac{a_{0,i}(x)n^i}{Q_0(n,x)} F_n(x) + \frac{a_{1,i}(x)n^i}{Q_1(n,x)} F_{n+1}(x) \right)$$

where $a_{0,i}(x)$ and $a_{1,i}(x)$ are polynomials

- May not always return the minimal order equations
- A lot faster than Chyzak's algorithm

Algo3: Reduction based Creative Telescoping[1]

Decompose derivatives $\partial_x^i(F_n(x))$ modulo the image of Δ_n:

- $F_n(x) = F_n(x)$
- $\partial_x(F_n(x)) = \frac{2(2n^2+1)}{3x} F_n(x) + \Delta_n(G_1)$
- $\partial_x^2(F_n(x)) = \frac{8n^2-3x^2-2}{3x^2} F_n(x) + \Delta_n(G_2)$

And find a $\mathbb{Q}(x)$-linear combination eliminating the term in $F_n(x)$:

$$x^2\partial_x^2(F_n(x)) - 2x\partial_x(F_n(x)) + (x^2+2)F_n(x) = \Delta_n(x^2 G_2 - 2xG_1)$$

which after summation gives

$$x^2\partial_x^2(S) - 2x\partial_x(S) + (x^2+2)S = 0$$

[1] For D-finite integrals: Bostan-Chyzak-Lairez-Salvy, van der Hoeven, Chen-Du-Kauers
For sums: (D-finite) van der Hoeven, (P-finite) Chen-Du-Kauers

Algo 3: Pseudocode

💡 Decompose derivatives $\partial_x^i(F_n(x))$ modulo the image of Δ_n

Require: a D-finite function $F_n(x)$
Ensure: a telescoper L and its associated certificate G

1: **for** $i = 0, 1, 2, \ldots$ **do**
2: Decompose $\partial_x^i(F) = R_i F + \Delta_n(G_i)$ with R_i "minimal"
3: **if** there is a $\mathbb{Q}(x)$-linear combination $\sum_{j \leq i} a_j R_j = 0$ **then**
4: **return** $\sum_{j \leq i} a_j \partial^j, \sum_{j \leq i} a_j G_j$
5: **end if**
6: **end for**

The algorithm generalizes to functions F with more parameters

Reduction of derivatives modulo $\mathrm{Im}(\Delta_n)$

Recall the equations:

$$-2x(2n+1)(n+1)^2\partial_x(F_n) + n^2x^2F_{n+1} + (n+1)^2(8n^2 - x^2 + 4n)F_n = 0$$
$$n^2(n+1)(2n+1)x^2F_{n+2} + (...)F_{n+1} + x^2(n+1)(2n+3)(n+2)^2F_n = 0$$

Using these equations it is possible to decompose $\partial_x(F_n)$ as follow:

$$\begin{aligned}\partial_x(F_n) &= \frac{n^2x}{2(2n+1)(n+1)^2}F_{n+1} + \frac{8n^2 - x^2 + 4n}{2x(2n+1)}F_n \\ &= \left(\frac{(n-1)^2x}{2n^2(2n-1)} + \frac{8n^2 - x^2 + 4n}{2x(2n+1)}\right)F_n \\ &\quad + \Delta_n\left(\frac{(n-1)^2x}{2n^2(2n-1)}F_n\right)\end{aligned}$$

It is possible to further reduce the coefficient in front of F_n modulo $\mathrm{Im}(\Delta_n)$.

Lagrange's identity

$L(f) = \sum_{i=0}^{r} a^i S_n^i(f) \quad \longleftrightarrow \quad L^*(f) = \sum_{i=0}^{r} a_i(n-i) S_n^{-i}(f)$

Lagrange's identity (Barrett, Dristy 1960)

Let $u(n), v(n)$ be two sequences and $L \in \mathbb{Q}(n,x)\langle S_n \rangle$ then

$$uL(v) - L^*(u)v = \Delta_n(P_L(u,v))$$

where P_L is linear in u and v.

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$$uL(v) - L^*(u)v = \Delta_n(P_L(u,v))$$

where P_L is linear in u and v.

Take $v = F$, $u \in \mathbb{Q}(n,x)$, and L minimal annihilating F, this identity gives

$$L^*(u)F = \Delta_n(-P_L(u,F))$$

Computing modulo $\mathrm{Im}(\Delta_n) \Leftrightarrow$ computing modulo $\mathrm{Im}(L^*)$

For all $R \in \mathbb{Q}(n,x)$

$$RF \in \mathrm{Im}(\Delta_n) \text{ if and only if } R \in L^*(\mathbb{Q}(n,x))$$

Reduction by a difference operator

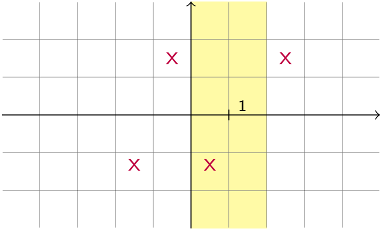
$$L^* = \sum_{i=0}^{r} p_i(n,x) S_n^{-i}$$

We want to define a $\mathbb{Q}(x)$-linear map $[.]$: $\mathbb{Q}(n,x) \to \mathbb{Q}(n,x)$ such that for all $R \in \mathbb{Q}(n,x)$

- $[R] - R \in L^*(\mathbb{Q}(n,x))$
- $[L^*(R)] = 0$

Reduction of poles

Assume $L^* = \sum_{i=0}^{r} p_i(n,x) S_n^{-i}$ has order $r = 2$ and $R \in \mathbb{Q}(n,x)$ has all its poles in $\mathbb{C}$ as a r.f. in n.



x : poles of R to be reduced by $\mathrm{Im}(L^*)$

Concentrate the poles in the yellow area

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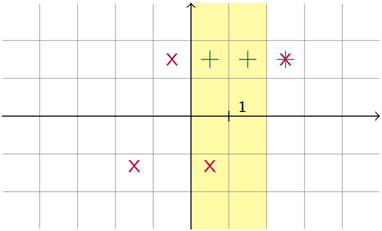

× : poles of R to be reduced by $\mathrm{Im}(L^*)$
$+$: poles of $L^*(1/(n-(1/2+i3/2))$
Reduction step:

$$R \leftarrow R - L^* \left(\frac{(\ldots)}{(n-(1/2+i3/2)^{\Box}} \right)$$

Reduction of poles

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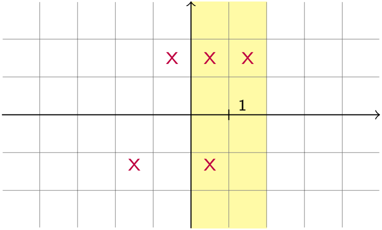

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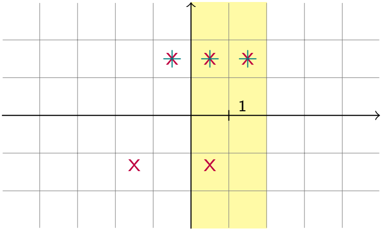

$\times$: poles of R to be reduced by $\mathrm{Im}(L^*)$
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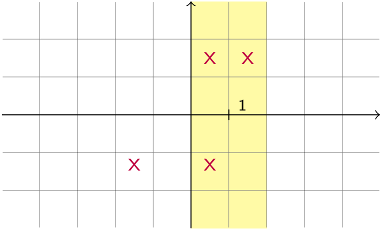

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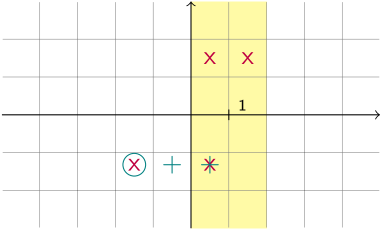

x : poles of R to be reduced by $\mathrm{Im}(L^*)$
$+$: poles of $L^*(1/(n-(-3/2-i4/3))$
◯ : not a pole of $L^*(1/(n-(-3/2-i4/3))$ because $1/(n-(-1/2+i3/2)$ is a singularity of p_0

Reduction of poles

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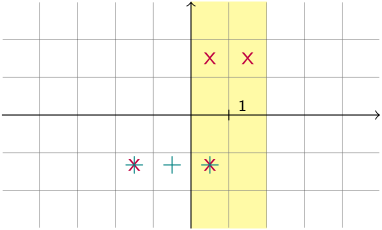

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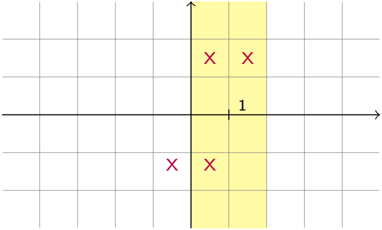

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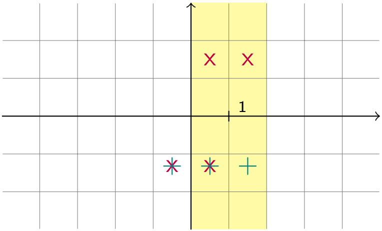

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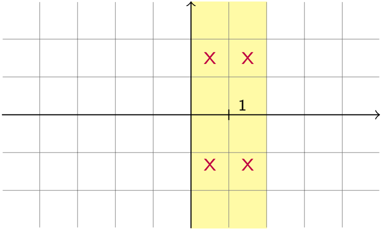

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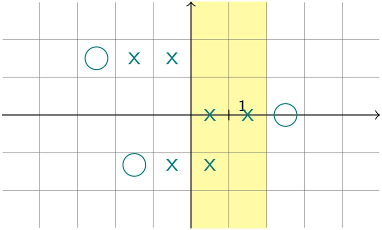
⚠ Is it enough ? No !

Strong reduction of poles

Let $[.]_w$ be the reduction procedure described previously



x : poles of $L^*(1/(n-\alpha))$, α root of $p_0(n)$ or $p_r(n-r)$ of order n_α
◯: not a pole because of a cancelation

$$E = Vect_{\mathbb{Q}(x)}\{[L^*(1/(n-\alpha)^i)]_w \mid \alpha, i \leq n_\alpha\}$$

Stong reduction: reduce $[R]_w$ modulo E

Reduction of polynomials

Similar (skipped)

Timing 1: (mostly) special functions

	HF-CT	HF-FCT	redctsum
21 easy examples	10.0s	9.2s	2.4s
eq. (1)	99s	50s	1.2s
eq. (2)	2138s	44s	13.8s
eq. (3)	63s	1.6s	39s
eq. (4)	4.5s	1.4s	61s
eq. (5)	$>$1h	3.2s(*)	$>$1h
eq. (6)	$>$1h	108s(*)	$>$1h
eq. (7)	$>$1h	$>$ 1h	1.2s

$$\sum_j \binom{m+x}{m-i+j} c_{n,j} \quad \text{where } c_{n,j} \text{ satisfies recurrences of order 2} \tag{1}$$

$$\sum_n C_n^{(k)}(x) C_n^{(k)}(y) \frac{u^n}{n!} \tag{2}$$

$$\sum_n J_n(x) C_n^k(y) \frac{u^n}{n!} \tag{3}$$

$$\sum_n \frac{(4n+1)(2n)!\sqrt{2\pi}}{n!^2 2^{2n} \sqrt{x}} J_{2n+1/2}(x) P_{2n}(u) \tag{4}$$

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$$\sum_n P_n(x)P_n(y)P_n(z) \tag{5}$$

$$\sum_k \frac{(a+b+1)_k}{(a+1)_k(b+1)_k} J_k^{(a,b)}(x)J_k^{(a,b)}(y) \tag{6}$$

$$\sum_y \frac{4x+2}{(45x+5y+10z+47)(45x+5y+10z+2)(63x-5y+2z+58)(63x-5y+2z-5)} \tag{7}$$

Timing 2: Gillis-Reznick-Zeilberger sequence

$$S_r = \sum_{k=0}^{n} \frac{(-1)^k (rn - (r-1)k)!(r!)^k}{(n-k)!^r k!}$$

Telescoper of order r and degree $r(r-1)/2$

	HF-CT	HF-FCT	redctsum
S_6	11s	64s	0.4s
S_7	32s	331s	0.9s
S_8	106s	1044s	2s
S_9	325s	3341s	5s
S_{10}	1035s	$>$1h	8s

Links

arxiv link: `https://arxiv.org/abs/2307.07216`
github link: `https://github.com/HBrochet/CreativeTelescoping`