Multivariate integration in D-modules

Hadrien Brochet,
Frédéric Chyzak, Pierre Lairez

Inria Saclay

Domosch∞l 2024

A symbolic integration problem

Let $I(t) = \int\limits_{\gamma} \frac{dx\,dy\,dz}{1-(1-xy)z-txyz(1-x)(1-y)(1-z)}$ (g.f. of Apéry numbers)

The objective is to compute a lin. ord. diff. eq. (LODE) for I:

$$t^2(t^2-34t+1)\frac{\partial^3 I}{\partial t^3}+3t(2t^2-51t+1)\frac{\partial^2 I}{\partial t^2}+(7t^2-112t+1)\frac{\partial I}{\partial t}+(t-5)I=0.$$

A symbolic integration problem

Let $I(t) = \int\limits_{\gamma} \frac{dx\,dy\,dz}{1-(1-xy)z-txyz(1-x)(1-y)(1-z)}$ (g.f. of Apéry numbers)

The objective is to compute a lin. ord. diff. eq. (LODE) for I:

$$t^2(t^2-34t+1)\frac{\partial^3 I}{\partial t^3}+3t(2t^2-51t+1)\frac{\partial^2 I}{\partial t^2}+(7t^2-112t+1)\frac{\partial I}{\partial t}+(t-5)I=0.$$

With this LODE it is possible to

1. compute a series expansion,
2. evaluate the integral numerically,
3. prove identities involving $I(t)$.

Algebra of differential operators

Let D be the algebra $\mathbb{K}(t)[\underline{x}]\langle \partial_t, \underline{\partial_x} \rangle$ where each pair of variables commutes except $\partial_i x_i = x_i \partial_i + 1$, $\partial_t t = t\partial_t + 1$

Algebra of differential operators

Let D be the algebra $\mathbb{K}(t)[\underline{x}]\langle\partial_t, \underline{\partial_x}\rangle$ where each pair of variables commutes except $\partial_i x_i = x_i\partial_i + 1$, $\partial_t t = t\partial_t + 1$

- It acts on C^∞ functions: $x_i \cdot f = x_i f$ and $\partial_i \cdot f = \frac{\partial f}{\partial x_i}$ (resp. t, ∂_t).

Algebra of differential operators

Let D be the algebra $\mathbb{K}(t)[\underline{x}]\langle\partial_t, \underline{\partial_x}\rangle$ where each pair of variables commutes except $\partial_i x_i = x_i\partial_i + 1$, $\partial_t t = t\partial_t + 1$

- It acts on C^∞ functions: $x_i \cdot f = x_i f$ and $\partial_i \cdot f = \frac{\partial f}{\partial x_i}$ (resp. t, ∂_t).
- It has a monomial basis as $\mathbb{K}(t)$-vector space: $(\underline{x}^\alpha \underline{\partial_x}^\beta \partial_t^\gamma)_{\alpha,\beta,\gamma}$.
- It has a notion of degree:

$$\deg\left(\sum_{\alpha,\beta,\gamma} c_{\alpha,\beta,\gamma}\underline{x}^\alpha \underline{\partial_x}^\beta \partial_t^\gamma\right) = \max\{|\alpha| + |\beta| + \gamma \mid c_{\alpha,\beta,\gamma} \neq 0\}.$$

- Any left (resp. right) ideal I of D has a finite Gröbner basis.

Holonomy

Holonomic function

A function f is holonomic if for each ∂_i and for ∂_t it satisfies a LODE with polynomial coefficients in $\mathbb{K}[t, x_1, \ldots, x_n]$.

Annihilator of f

The set $\operatorname{Ann}(f) \overset{\text{def}}{=} \{L \in D \mid L \cdot f = 0\}$ is a left ideal

Link between functions and operators

f is characterized by its annihilator: $D \cdot f \simeq D / \operatorname{Ann}(f)$.

Integration using Creative telescoping

Assumption: the integral of a derivative is zero

For any $b \in D$ and $i \in [\![1, n]\!]$, $\int_\gamma \partial_i b \cdot f \, \underline{dx} = 0$.

Creative telescoping

Find $\ell \in \mathbb{N}$, $a_1, \ldots, a_\ell \in \mathbb{K}(t)$ and $b_1, \ldots, b_n \in D$ such that

$$\Big(a_\ell(t)\partial_t^\ell + \cdots + a_1(t)\partial_t + a_0(t)\Big) \cdot f = \sum_{i=1}^{n} \partial_i b_i \cdot f$$

after integration we get

$$\Big(a_\ell(t)\partial_t^\ell + \cdots + a_1(t)\partial_t + a_0\Big) \cdot \int_\gamma f \, \underline{dx} = 0$$

Algebraic analog of creative telescoping

Creative telescoping

Find $\ell \in \mathbb{N}$, $a_1, \dots, a_\ell \in \mathbb{K}(t)$ and $b_1, \dots, b_n \in D$ such that

$$\left(a_\ell(t)\partial_t^\ell + \cdots + a_1(t)\partial_t + a_0(t)\right) \cdot f = \sum_{i=1}^{n} \partial_i b_i \cdot f$$

Algebraic analog

Compute the telescoping ideal $\mathcal{I}(f) = \left(\mathrm{Ann}(f) + \sum_{i=1}^{n} \partial_i D\right) \cap \mathbb{K}(t)\langle\partial_t\rangle$

Algebraic analog of creative telescoping

Creative telescoping

Find $\ell \in \mathbb{N}$, $a_1, \ldots, a_\ell \in \mathbb{K}(t)$ and $b_1, \ldots, b_n \in D$ such that

$$\left(a_\ell(t)\partial_t^\ell + \cdots + a_1(t)\partial_t + a_0(t)\right) \cdot f = \sum_{i=1}^{n} \partial_i b_i \cdot f$$

Algebraic analog

Compute the telescoping ideal $\mathcal{I}(f) = \left(\operatorname{Ann}(f) + \sum_{i=1}^{n} \partial_i D\right) \cap \mathbb{K}(t)\langle\partial_t\rangle$

Theorem

f holonomic $\implies \mathcal{I}(f)$ non-zero.

Takayama's algorithm for integration[1]

Difficulty: $\mathrm{Ann}(f) + \sum_{i=1}^{n} \partial_i D$ is not a left D-ideal.

But it is an infinite rank left $\mathbb{K}(t)\langle \partial_t, \underline{\partial_x} \rangle$-module.

[1] Takayama 1990, (improved in Chyzak-Salvy98 but not presented today)

Takayama's algorithm for integration[1]

Difficulty: $\mathrm{Ann}(f) + \sum_{i=1}^{n} \partial_i D$ is not a left D-ideal.

But it is an infinite rank left $\mathbb{K}(t)\langle \partial_t, \underline{\partial_x} \rangle$-module.

Takayama's algorithm

Input: $\mathrm{Ann}(f)$

Output: non-trivial subset of $(\mathrm{Ann}(f) + \sum_{i=1}^{n} \partial_i D) \cap \mathbb{K}(t)\langle \partial_t \rangle$

- Take generators of $\mathrm{Ann}(f) + \sum_{i=1}^{n} \partial_i D$ up to a certain degree.
- Eliminate $\underline{x}$ and $\underline{\partial_x}$ via Gröbner basis computation
- If we find an operator without $\underline{x}$ and $\underline{\partial_x}$ return it otherwise increase the degree bound and start again.

[1] Takayama 1990, (improved in Chyzak-Salvy98 but not presented today)

Creative telescoping using reductions

Goal: Find a LODE for $\int_\gamma a \cdot f \, \underline{dx}$ for $a \in D$ and f holonomic

Creative telescoping using reductions

Goal: Find a LODE for $\int_\gamma a \cdot f \, \underline{dx}$ for $a \in D$ and f holonomic

Subgoal: Construct a $\mathbb{K}(\underline{t})$-linear map $[.] : D \to D$ s.t.

- $a - [a] \in \mathrm{Ann}(f) + \sum_{i=1}^{n} \partial_i D$ (reduction)
- $[a] = 0$ iff $a \in \mathrm{Ann}(f) + \sum_{i=1}^{n} \partial_i D$ (canonical form)

Creative telescoping using reductions

Goal: Find a LODE for $\int_\gamma a \cdot f \, \underline{dx}$ for $a \in D$ and f holonomic

Subgoal: Construct a $\mathbb{K}(\underline{t})$-linear map $[.] : D \to D$ s.t.

- $a - [a] \in \operatorname{Ann}(f) + \sum_{i=1}^{n} \partial_i D$ (reduction)
- $[a] = 0$ iff $a \in \operatorname{Ann}(f) + \sum_{i=1}^{n} \partial_i D$ (canonical form)

Creative telescoping algorithm:[1]

$p_0 \leftarrow [a]; \quad \ell \leftarrow 0$

while there is no $\mathbb{K}(t)$-linear relation $\sum_{i=0}^{\ell} \lambda_i p_i = 0$

$\qquad p_{\ell+1} \leftarrow [\partial_t p_\ell]; \quad \ell \leftarrow \ell + 1$

return $\sum_{i=0}^{\ell} \lambda_i \partial_t^i$

[1] Bostan, Chen, Chyzak, Li, Xin

A first "naïve" reduction

Algorithm $[.]_{GD}$:

repeat
    $a \leftarrow RRem(a, \sum_{i=1}^{n} \partial_i D)$
    $a \leftarrow LRem(a, \mathrm{Ann}(f))$
until no term in a can be further reduced
return a

A first "naïve" reduction

Algorithm $[.]_{GD}$:

```
repeat
    a ← RRem(a, ∑_{i=1}^n ∂_i D)
    a ← LRem(a, Ann(f))
until no term in a can be further reduced
return a
```

The Griffiths-Dwork reduction is a special case of the above algorithm.

Theorem (Dwork 62, Griffiths 69)

If $f = e^q$ with $q \in \mathbb{K}(t)[\underline{x}]$ homogeneous and $V(q)$ is smooth then $[.]_{GD}$ is a canonical form.

A more effective reduction

Precomputation (reminiscent of Takayama's algorithm)

$B_\sigma \leftarrow \emptyset$
for each gen. g of $\mathrm{Ann}(f) + \sum_{i=1}^{n} \partial_i D$ as $\mathbb{K}(t)\langle \partial_t, \underline{\partial_x} \rangle$-module s.t. $\deg_{\underline{x}}(g) \leq \sigma$
    $B_\sigma \leftarrow B_\sigma \cup \{[g]_{GD}\}$
$B_\sigma \leftarrow GB(B_\sigma)$ over $\mathbb{K}(t)\langle \partial_t \rangle$
return B_σ

Algorithm $[.]_T^\sigma$:

return $\mathrm{LRem}([a]_{GD}, B_\sigma)$ over $\mathbb{K}(t)\langle \partial_t \rangle$

A more effective reduction

Precomputation (reminiscent of Takayama's algorithm)

$B_\sigma \leftarrow \emptyset$
for each gen. g of $\mathrm{Ann}(f)$ as $\mathbb{K}(t)\langle\partial_t, \underline{\partial_x}\rangle$-module s.t. $\deg_{\underline{x}}(g) \leq \sigma$
    $B_\sigma \leftarrow B_\sigma \cup \{[g]_{GD}\}$
$B_\sigma \leftarrow GB(B_\sigma)$ over $\mathbb{K}(t)\langle\partial_t\rangle$
return B_σ

Algorithm $[.]_T^\sigma$:

return $\mathrm{LRem}([a]_{GD}, B_\sigma)$ over $\mathbb{K}(t)\langle\partial_t\rangle$

A more effective reduction

$I = \{g \in \mathrm{Ann}(f) \mid \mathrm{lm}(g) \in \mathbb{K}[\underline{x}]\langle\partial_t\rangle\}$

Precomputation (reminiscent of Takayama's algorithm)

$B_\sigma \leftarrow \emptyset$
for each gen. g of $\mathrm{Ann}(f)$ as $\mathbb{K}(t)\langle\partial_t, \underline{\partial_x}\rangle$-module
  s.t. $\deg_{\underline{x}}(g) \leq \sigma$
  if $\mathrm{lm}(g)$ is not reducible by I
    $B_\sigma \leftarrow B_\sigma \cup \{[g]_{GD}\}$
$B_\sigma \leftarrow GB(B_\sigma)$ over $\mathbb{K}(t)\langle\partial_t\rangle$
return B_σ

Algorithm $[.]_T^\sigma$:

return $\mathrm{LRem}([a]_{GD}, B_\sigma)$ over $\mathbb{K}(t)\langle\partial_t\rangle$

A more effective reduction

$I = \{g \in \mathrm{Ann}(f) \mid \mathrm{Im}(g) \in \mathbb{K}[\underline{x}]\langle \partial_t \rangle\}$

Precomputation (reminiscent of Takayama's algorithm)

$B_\sigma \leftarrow \emptyset$
for each gen. g of $\mathrm{Ann}(f)$ as $\mathbb{K}(t)\langle \partial_t, \underline{\partial_x} \rangle$-module s.t. $\deg_{\underline{x}}(g) \leq \sigma$
    if $\mathrm{Im}(g)$ is not reducible by I
        $B_\sigma \leftarrow B_\sigma \cup \{[g]_{GD}\}$
$B_\sigma \leftarrow GB(B_\sigma)$ over $\mathbb{K}(t)\langle \partial_t \rangle$
return B_σ

Algorithm $[.]^\sigma_T$:

return $\mathrm{LRem}([a]_{GD}, B_\sigma)$ over $\mathbb{K}(t)\langle \partial_t \rangle$

Theorem

For any $a \in \mathrm{Ann}(f) + \sum_{i=1}^n \partial_i D$, $[a]^\sigma_T = 0$ for σ large enough.

Conclusion and future works

We have:

- A new algorithm for multivariate integration in Weyl algebras
- An optimized implementation for 1 param. (CRT + evaluation/interpolation)
- Application to other fields (e.g. combinatorics)

Next:

- Test this algorithm with Feynman integrals
- Design an integration algorithm for rational Weyl algebras

Application: counting k-regular graphs

k-regular graph: every vertex has degree k

Problem statement

Let $c_n^{(k)}$ be number of k-regular graphs on n vertices.
We want to compute a LODE for $\sum_{n=0}^{\infty} c_n^{(k)} t^n$ for $k = 1, 2, 3, \ldots$

Previous work

- 1959-(Read): up to $k = 3$
- $\approx$ 1959-(McKay, Wormald): $k = 4$
- 2024-(Chyzak, Mishna): up to $k = 7$.

Application: counting k-regular graphs

$\rightsquigarrow$ Using different tools (scalar product of symmetric functions, Laplace transform ...), we obtain

$$\sum_{n=0}^{\infty} c_n^{(k)} t^n = \operatorname{res}_{p_1,\ldots,p_k} F(t, p_1, \ldots, p_k)$$

where F is implicitly represented by an ideal $I \subset \mathbb{K}(t)[\underline{p}]\langle \partial_t, \underline{\partial_p} \rangle$ satisfying for any $L \in I$, $\operatorname{res}_{\underline{p}} L(F) = 0$.

Application: counting k-regular graphs

$\rightsquigarrow$ Using different tools (scalar product of symmetric functions, Laplace transform ...), we obtain

$$\sum_{n=0}^{\infty} c_n^{(k)} t^n = \mathrm{res}_{p_1,\ldots,p_k} F(t, p_1, \ldots, p_k)$$

where F is implicitly represented by an ideal $I \subset \mathbb{K}(t)[\underline{p}]\langle \partial_t, \underline{\partial_p} \rangle$ satisfying for any $L \in I$, $\mathrm{res}_{\underline{p}} L(F) = 0$.

- Most C.T. algorithms do not apply (they need rat coeffs)
- Computation for $k = 7$ done in 6 min (vs 25 weeks for Chyzak-Mishna)
- It validates the correction and efficiency of our algorithm