Faster symbolic integration in D-modules

Hadrien Brochet,
Frédéric Chyzak, Pierre Lairez

Inria Saclay

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The problem of symbolic integration

Let $I(t) = \int\limits_{\gamma} \frac{dx\,dy\,dz}{1-(1-xy)z-txyz(1-x)(1-y)(1-z)}$.

The objective is to compute a LDE for I:

$$t^2(t^2-34t+1)\frac{\partial^3 I}{\partial t^3}+3t(2t^2-51t+1)\frac{\partial^2 I}{\partial t^2}+(7t^2-112t+1)\frac{\partial I}{\partial t}+(t-5)I=0.$$

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Two families of algorithms to solve it:

1. rational Creative Telescoping algorithms
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But D-modules algorithms are more expressive.
Goal: reduce the gap

Definition of the Weyl Algebra

The Weyl algebra W_n is the algebra $\mathbb{K}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ with the commutation rules $\partial_i x_i = x_i \partial_i + 1$.

- It acts on C^∞ functions: $x_i \cdot f = x_i f$ and $\partial_i \cdot f = \frac{\partial f}{\partial x_i}$.
- It has a monomial basis as a $\mathbb{K}$-vector space: $(\underline{x}^\alpha \underline{\partial}^\beta)_{\alpha,\beta}$.
- It has a notion of degree:

$$\deg\Big(\sum_{\alpha,\beta} c_{\alpha,\beta} \underline{x}^\alpha \underline{\partial}^\beta\Big) = \max\{|\alpha| + |\beta| \mid c_{\alpha,\beta} \neq 0\}.$$

- finitely generated W_n-modules admit Gröbner bases.
- f. g. W_n-modules have a dimension given by its Hilbert serie.

Holonomy

Holonomic function

A function f is holonomic if it satisfies a LDE with polynomial coefficients w.r.t. each of its variables.

Annihilator of f

It is the left ideal defined as $\text{Ann}(f) \stackrel{\text{def}}{=} \{L \in W_n \mid L \cdot f = 0\}$.

Holonomic Ideal

A non-trivial left ideal I of W_n is holonomic if $dim(W_n/I) = n$. (Bernstein's inequality : $n \leq dim(W_n/I) \leq 2n$).

Algebraic equivalent of integration

Integration ideal w.r.t x_n

The integration ideal of f is $\mathrm{II}(f) = (\mathrm{Ann}(f) + \partial_n W_n) \cap W_{n-1}$.

Explanation:

Let γ be a loop and $P = A + \partial_n B \in \mathrm{II}(f)$, then

$$P \cdot \int_\gamma f dx_n = \int_\gamma P \cdot f dx_n = \int_\gamma \underbrace{A \cdot f}_{=0 \text{ as } A \in \mathrm{Ann}(f)} dx_n + \underbrace{\int_\gamma \partial_n B \cdot f dx_n}_{=0 \text{ as } \gamma \text{ is a loop}} = 0.$$

Thus

$$P \in \mathrm{Ann}\left(\int_\gamma f dx_n\right).$$

Comparison with Creative Telescoping

Let $W_n^{\text{rat}} = \mathbb{K}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$

Creative telescoping algorithms[1] work in W_n^{rat} to compute

$$\text{II}^{rat}(f) = (\text{Ann}^{\text{rat}}(f) + \partial_n W_n^{\text{rat}}) \cap W_{n-1}^{\text{rat}}.$$

Advantages over D-modules

- CT algorithms are faster in practice
- Gröbner basis w.r.t an order on n variables only.
- The ideal $\text{II}^{\text{rat}}(f) \cap W_{n-1}$ might be larger that $\text{II}(f)$

But it is not as general (see next slide).

[1] Zeilberger 1989, Chyzak 2000, Koutschan 2010, Bostan-Chyzak-Lairez-Salvy 2018

Integration over semi-algebraic set[1] in W_n

The Heaviside function $H : x \mapsto 1$ if $x > 0$ and 0 otherwise is holonomic in W_n and satisfies

$$x\partial_x \cdot H = 0.$$

in the sense of Schwartz distributions. But in W_n^{rat} it is equivalent to

$$\partial_x \cdot H = 0$$

which would mean that H is a constant function (contradiction).

[1] Oaku-Shiraki-Takayama 2003, Oaku 2013

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By closure properties, indicator functions of semi-algebraic sets are holonomic, which permits us to compute integrals like

$$\int_{x^3-y^2\geq 0} e^{-t(x^2+y^2)} dx\, dy = \int_{\mathbb{R}^2} e^{-t(x^2+y^2)} \mathbb{1}_{\{x^3-y^2\geq 0\}} dx\, dy.$$

[1] Oaku-Shiraki-Takayama 2003, Oaku 2013

Takayama's algorithm for integration[1]

Main difficulty: $\mathrm{Ann}(f) + \partial_n W_n$ is not stable by x_n.
Solution: write $W_n = \sum_{i=0}^{+\infty} W_{n-1}[\partial_n] x_n^i$ and compute GB over W_{n-1}.

Input $\mathrm{Ann}(f) = (g_1, \dots, g_r)$ a holonomic ideal
Output GB of a holonomic subideal of $(\mathrm{Ann}(f) + \partial_n W_n) \cap W_{n-1}$

$s \leftarrow 0$
repeat
  $s \leftarrow s + 1$
  $G_s \leftarrow \{x_n^i g_j \mid \deg_{x_n}(g_j) + i \leq s\}$ (generators of $\mathrm{Ann}(f)$ of degree at most s in x_n)
  $G_s \leftarrow G_s$ modulo $\partial_n W_n$
  $G_s \leftarrow$ Gröbner basis of G_s w.r.t. $\preccurlyeq$ eliminating x_n
  $\tilde{G}_s \leftarrow G_s \cap W_{n-1}$ (keep only operators without x_n)
until $W_{n-1}\tilde{G}_s$ is holonomic
return $\tilde{G}_s$

[1] Takayama 1998, Chyzak-Salvy 1998

Variants of the algorithm

$(W_{n-1}\tilde{G}_s)_s$ is a non-decreasing stationary sequence of ideals.

1. A bound on s is the maximal integer root of the b-function[1].

   More costly but guarantees to compute the full ideal.

2. Relax the Gröbner basis condition on the output and return as soon as we can garantee to have a holonomic subideal of II.

   Less costly but we might find LDE of larger order.

[1] Oaku-Takayama 1998

Integration of rational functions coming from the study of small step walks in the quarter plane[1]

- Maple : Chyzak's algorithm (CT)
- Singular: *b*-function criterion
- Julia (our implementation) : hol-GB et hol-fast

For example SSW2 is:

$$\int \frac{-x^2y^2 + x^2 + y^2 - 1}{tx^2y^2 + tx^2 + ty^2 - xy + t} dxdy$$

	CT	Singular		Julia-hol-GB		Julia-hol-fast		
	Both	Ann	Int	Ann	Int	Ann	Int	minimality
SSW1	1.23	$> 24h$	?	7798.62	315.33	180.68	3.11	yes
SSW2	1.03	$> 24h$	?	54.89	138.95	7.4	3.88	yes
SSW3	2.36	71.0	11.0	295.03	1091.45	26.52	38.07	yes
SSW4	4.04	$> 24h$	?	$> 24h$	?	319.06	79.42	yes
SSW5	0.96	$> 24h$	?	53857.47	1945.71	24.07	8.56	yes
SSW6	5.16	$> 24h$	?	$> 24h$	?	12084.26	64.07	no
SSW7	1.28	$> 24h$	?	$> 24h$	?	104.39	20.23	yes
SSW8	3.78	$> 24h$	?	$> 24h$	?	9321.11	102.15	no
SSW10	5.45	$> 24h$	?	$> 24h$	?	41768.99	558.54	no
SSW11	1.21	$> 24h$	?	73839.69	591.41	158.9	10.29	yes
SSW13	3.51	$> 24h$	?	$> 24h$	?	5587.42	201.58	no
SSW15	1.03	122.0	1.0	607.44	233.37	25.51	3.11	yes
SSW17	3.27	$> 24h$	?	$> 24h$	?	136.21	21.91	yes
SSW18	30.4	$> 24h$	?	$> 24h$	?	15863.88	849.91	yes

[1] Bostan-Chyzak-van Hoeij-Kauers-Pech 2017

Annihilators in W_n are hard to compute

Let $R = 1/(y^2 - x^3)$ then

$$\mathrm{Ann}^{\mathrm{rat}}(R) = W_n^{\mathrm{rat}}(\underbrace{(y^2 - x^3)\partial_x - 3x^2}_{g_1}) + W_n^{\mathrm{rat}}(\underbrace{(y^2 - x^3)\partial_y + 2y}_{g_2}).$$

But $W_n g_1 + W_n g_2$ is not holonomic !

Observe that $g_3 \in Ann^{\mathrm{rat}}(R) \setminus W_n g_1 + W_n g_2$:

$$g_3 \overset{def}{=} 2ydx + 3x^2 dy = \frac{1}{y^2 - x^3}(2yg_1 + 3x^2 g_2)$$

Computation of $\mathrm{Ann}(f)$ using saturation

Saturation ideal

Let I be a left ideal of W_n and p be a polynomial then

$$I_{\mathrm{sat}} = \{L \in W_n \mid \exists j \; p^j L \in I\}.$$

Proposition

Let G be a Gröbner basis of $\mathrm{Ann}^{\mathrm{rat}}(f)$ with polynomial coefficients and p be the lcm of the leading coefficients of G. Then

$$\mathrm{Ann}(f) = (W_n G)_{\mathrm{sat}}.$$

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Lemma: Computation of I_{sat}

Let T be a new variable with the commutation $\partial_i T = T\partial_i - \frac{\partial p}{\partial x_i} T^2$

$$I_{\mathrm{sat}} = (W_n[T]I + W_n[T](pT - 1)) \cap W_n.$$

Saturation algorithm

Difficulty: if $\preccurlyeq$ eliminates T, we don't have $\mathrm{lm}(ab) = \mathrm{lm}(a)\,\mathrm{lm}(b)$.
Solution: write $W_n[T] = \sum_{i=0}^{+\infty} W_n T^i$.

Input $\mathrm{Ann}^{\mathrm{rat}}(f) = (g_1, \dots, g_r)$ and p as before
Output GB of a holonomic subideal of $\mathrm{Ann}(f)$

$g_0 \leftarrow pT - 1$
repeat
  $s \leftarrow s + 1$
  $G_s \leftarrow \{T^i g_j \mid i \leq s\}$
  (generators of $W_n[T]I + W_n[T](pT - 1)$ with $\deg_T \leq s$)
  $G_s \leftarrow$ Gröbner basis of G_s in W_n w.r.t. $\preccurlyeq$ eliminating T
  $\tilde{G}_s \leftarrow G_s \cap W_n$ (keep only operators without T)
until $W_n \tilde{G}_s$ is holonomic
return $\tilde{G}_s$

Conclusion

- Faster algorithms by earlier termination (speed/minimality trade-off).
- A new algorithm for computing saturation in W_n.
- Performance are still unsatisfactory.