Faster Integration of D-modules

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Notations

- $W_n \coloneqq \mathbb{K}[x_1,\dots,x_n]\langle\partial_1,\dots,\partial_n\rangle$ — the nth Weyl algebra over a field $\mathbb{K}$
- $\operatorname{Ann}(f) \coloneqq \{L \in W_n \mid L\cdot f = 0\}$ — the annihilator of (e.g.) a C^∞ function f
- $\boldsymbol{\partial} M \coloneqq \sum_{i=1}^n \partial_i M$ — a notation for any set M

The problem of integration of D-modules

A function f is represented by its annihilator via the following quotient of left W_n-modules:

$$f \quad \rightsquigarrow \quad W_n\cdot f \quad \simeq \quad W_n/\operatorname{Ann}(f)$$

The integral of f over a cycle γ is represented by the same module up to derivatives:

$$\int_\gamma f(\mathbf{x})\mathbf{dx} \quad \rightsquigarrow \quad \frac{W_n\cdot f}{\boldsymbol{\partial} W_n\cdot f} \quad \simeq \quad \frac{W_n}{\operatorname{Ann}(f)+\boldsymbol{\partial} W_n}$$

$\operatorname{Ann}(f)$ is a left W_n-module; $\boldsymbol{\partial} W_n$ is a right W_n-module.

Theorem (Kashiwara)

If $\operatorname{Ann}(f)$ is holonomic, then $W_n/(\operatorname{Ann}(f)+\partial W_n)$ is a finite-dimensional vector space.

Problem: How to compute a basis of $W_n/(\operatorname{Ann}(f)+\boldsymbol{\partial} W_n)$?

Difficulties

1. $W_n/(\operatorname{Ann}(f)+\boldsymbol{\partial} W_n)$ is not a W_n-module but *only* a $\mathbb{K}$-vector space.
2. $W_n/(\operatorname{Ann}(f)+\boldsymbol{\partial} W_n)$ is the quotient of two infinite-dimensional $\mathbb{K}$-vector spaces.

Takayama's algorithm [2]

The Weyl algebra W_n admits as infinite dimensional $\mathbb{K}$-vector space the filtration

$$F_q = \bigoplus_{|\alpha|+|\beta|\le q} \mathbb{K}\cdot \mathbf{x}^\alpha \boldsymbol{\partial}^\beta.$$

Takayama's algorithm (simplified)

Iteratively compute a basis of the quotient

$$F_q/(\operatorname{Ann}(f)\cap F_q + \boldsymbol{\partial} F_{q-1})$$

of finite-dimensional $\mathbb{K}$-vector spaces.

In Takayama's paper not every variable ∂_i are eliminated.

Termination criterion

Oaku and Takayama gave a bound on q to get a basis of the whole vector space $W_n/(\operatorname{Ann}(f)+\boldsymbol{\partial} W_n)$ based on the largest root of the b-function of the ideal $\operatorname{Ann}(f)$ associated to a weight vector $(w,-w)$ and a filtration with respect to this weight vector.

New algorithm by reduction

Idea

The vector space $\operatorname{Ann}(f)+\boldsymbol{\partial} W_n$ is the sum of a left and a right W_n-module. We can still use some of this hybrid structure to speed up computations.

The idea is to define a $\mathbb{K}$-linear map on W_n that reduces alternatively modulo the left and the right module. This reduction is then used in the new algorithm.

Reduction procedure $[.] : W_n \mapsto W_n$

```
repeat
    a ← a mod ∂W_n
    a ← a mod Ann(f)
until no term in a can be further reduced
return a
```

Applying the reduction $[.]$ to the vector space F_q *can* greatly reduce its dimension.

New Algorithm

Iteratively compute a basis of the quotient of finite-dimensional $\mathbb{K}$-vector spaces

$$\frac{[F_q]}{[\operatorname{Ann}(f)\cap F_q + \boldsymbol{\partial} F_{q-1}]} \quad \simeq \quad \frac{[F_q]}{[\operatorname{Ann}(f)\cap F_q]}$$

Integration of parametric integrals

Let $f(t,\mathbf{x})$ be a function depending on a new parameter t and let $\mathbb{K}(t)$ be the new base field. We assume that the derivation ∂_t defines a morphism of W_n-module onto $W_n/\operatorname{Ann}(f)$.

Problem 2: Compute a LDE for the parametric integral $\int_\gamma f(t,x)dx$.

This is solved by decomposing the successive derivatives $\partial_t^\ell(\bar{1})$ in the basis of $W_n/(\operatorname{Ann}(f)+\boldsymbol{\partial} W_n)$ until a linear relation is found.

Finding a LDE satisfied by the integral $I(t)$

```
p_0 ← 1;  ℓ ← 0
B ← a basis of W_n/(Ann(f) + ∂W_n)
while there exists no K(t)-linear relation Σ_{i=0}^{ℓ} λ_i p_i = 0
    p_{ℓ+1} ← Decomposition of ∂_t^{ℓ+1}(1̄) in the basis B
    ℓ ← ℓ + 1
return Σ_{i=0}^{ℓ} λ_i ∂_t^i
```

In general the derivation ∂_t is defined on a module of the form W_n^r/S instead of $W_n/\operatorname{Ann}(f)$.

Application to counting k-regular graphs

Let $c_n^{(k)}$ be the number of k-regular graphs on n vertices, that is, the number of graphs with exactly k neighbors.

Goal: Given k compute a LDE for the generating series $\sum_n c_n^k t^n$.

The first step is to express this generating series as the formal residue of a power series $F(t,x_1,\dots,x_k)$:

$$\sum_{n=0}^{\infty} c_n^{(k)} t^n = \operatorname{res}_{\mathbf{x}} F(t,\mathbf{x})$$

represented by a derivation ∂_t and an ideal $I \subset W_n$ satisfying $\operatorname{res}_{\mathbf{x}} L\cdot F = 0$ for any $L \in I$. The second step is to apply our algorithm to compute the LDE. We managed to compute a LDE for k up to 8.

Timings

k	2	3	4	5	6	7	8
Tak-Macaulay2	0.02s	1.7s	535s	>90m	-	-	-
Tak-Singular	<1s	<1s	25s	>90m	-	-	-
Our alg*	7.2s	7.6s	8.7s	7.9s	8.5s	363s	7h28min

Fig. 1: Computation time of a LDE satisfied by the generating series of k-regular graphs for different software and values of k.

* Our algorithm is implemented in Julia and is available on github [1]. Since Julia uses a JIT compilation mode, we included the compilation time of approximately 7.5 seconds in the timings. The timings were recorded on a personal laptop.

Do you recognise this graph ?

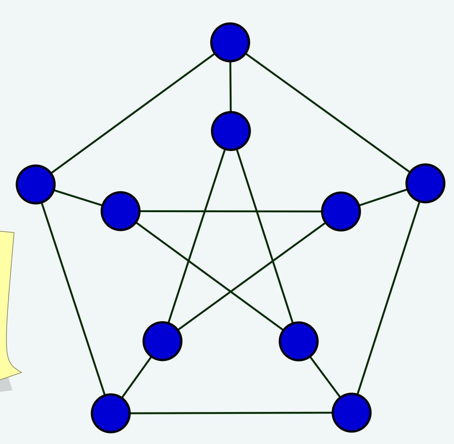

Fig. 2: A 3-regular graph

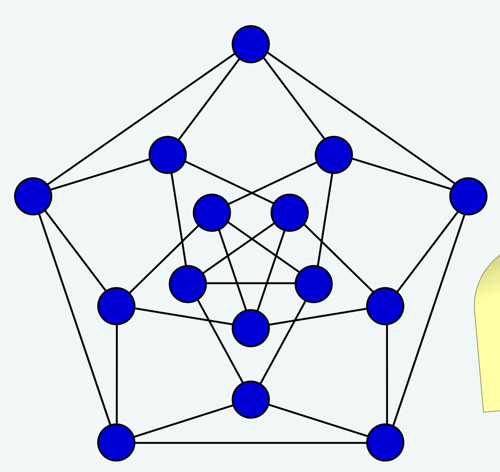

Fig. 3: A 4-regular graph

What about this one ?

References

[1] H. Brochet. Repository of the MultivariateCreativeTelescoping.jl package. https://github.com/HBrochet/MultivariateCreativeTelescoping.jl.

[2] N. Takayama. An algorithm of constructing the integral of a module–an infinite dimensional analog of gröbner basis. In *Proceedings of the International Symposium on Symbolic and Algebraic Computation*, ISSAC '90, page 206–211, New York, NY, USA, 1990. Association for Computing Machinery.